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A note on affine functions on convex sets.

by

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A note on affine functions on convex sets.

The purpose of this note is to study the prescribtion of boundary values for continuous affine functions on a compact convex subset K of a locally convex Hausdorff topological vectorspace E. Specifically, we shall prove the following:

Theorem. *Let K be a metrizable convex compact set, and let f be a continuous and bounded real valued function defined on the extreme boundary $\partial_e K$. The function f can be extended to a continuous affine function on K if and only if the following two requirements are satisfied:*

(i) *The envelopes $\bar{f}$ and $\underline{f}$ are continuous on $\partial_e K$*

(ii) $\int f d\nu = 0$ *for all affine dependeces ν on $\partial_e K$.*

Proof. 1) Sufficiency. The upper envelope $\bar{f}$ of f is the least upper semi-continuous function on K which majorizes f on its domain of definition, i.e. on $\partial_e K$. Most standard results on envelopes are stated for functions defined on the whole set K, and so it is relevant to observe that $\bar{f} = \bar{f}_1$ where f_1 is the following u.s.c. extension of f to K:

$$(1) \qquad f_1(x) = \begin{cases} \limsup\limits_{y \in \partial_e K,\, y \to x} f(y) & \text{for } x \in \overline{\partial_e K} \\ \alpha & \text{for } x \notin \overline{\partial_e K}, \end{cases}$$

where α is some constant such that $\alpha \leq f(x)$ for all $x \in \partial_e K$.

By a theorem of M. Herve [5]. $g(x) = \bar{g}(x)$ for $x \in \partial_e K$, provided that g is a continuous function on K. It is not difficult to verify that this result subsists if g is allowed to be u.s.c. (but not l.s.c.). Hence we shall have

(2) $$\bar{f}(x) = \bar{f}_1(x) = f_1(x) = f(x),$$

for every $x \in \partial_e K$.

The lower envelope $\underline{f}$ of f is characterized dually as the greatest l.s.c. convex minorant of f. Clearly we shall have $\underline{f} = \underline{f}_2$, where f_2 is defined by the dual formula to (1). Proceeding as above, we obtain the equivalent of (2) with f_2 in the place of f_1 and lower envelopes in the place of upper envelopes. Hence

(3) $$f(x) = \bar{f}(x) = \underline{f}(x) \quad \text{for} \quad x \in \partial_e K.$$

By the assumption (i), f may be extended to a continuous function $\tilde{f}$ on $\overline{\partial_e K}$, such that

(4) $$\tilde{f}(x) = \bar{f}(x) = \underline{f}(x) \quad \text{for} \quad x \in \overline{\partial_e K}.$$

If μ_1 and μ_2 are two members of the set $\mathcal{M}_x^+$ of all (Radon-)probability measures concentrated on $\partial_e K$ and with barycenter x, then the difference $\mu_1 - \mu_2$ is an affine dependence on $\partial_e K$. (For the definition of an affine dependence on $\partial_e K$ cf. [2, p.98]). It follows by the assumption (ii), that there exists a real valued function h on K such that

(5) $$h(x) = \int f d\mu, \qquad \mu \in \mathcal{M}_x^+.$$

It is easily verified that h is an affine extension of f. In fact h <u>is an extension of</u> $\tilde{f}$, for if $\mu \in \mathcal{M}_x^+$, then generally

$$\underline{f}(x) \leq \int f d\mu \leq \bar{f}(x).$$

The next step will be to prove that h is of the first Baire class.

It is well known that in the general (not necessarily metrizable) case, every upper envelope $\bar{f}$ is pointwise limit of some descending net of continuous concave functions. (The proof is based on Hahn-Banach separation in the product space

$E \times \mathbb{R}$, and it can be found e.g. in [4], and in [6, ch.3].)
In the case of a metrizable convex compact set one may use second countability to modify the proof of this fact such as to yield a descending <u>sequence</u>.

Now assume $\{g_n\}$ to be a sequence of continuous concave functions such that

(6) $$g_n \searrow \bar{f}.$$

By a known result (cf. e.g. [4], the lower envelope of any of the functions g_n may be characterized as follows:

(7) $$\underline{g}_n(x) = \inf\left\{\int g_n d\mu \,\middle|\, \mu \in \mathfrak{M}_x^+\right\}.$$

For any measure μ in $\mathfrak{M}_x^+$:

(8) $$\int g_n d\mu \geqq \int f d\mu = h(x)$$

Hence it follows that

(9) $$\underline{g}_n(x) \geqq h(x), \qquad n = 1,2,\ldots .$$

Again let μ be an arbitrary measure in $\mathfrak{M}_x^+$. By virtue of (2), (6), (7) and by the Monotone Convergence Theorem

(10) $$h(x) = \int f d\mu = \lim_{n\to\infty} \int g_n d\mu \geqq \lim_{n\to\infty} \underline{g}_n(x).$$

Hence by (8) and (10)

(11) $$\lim_{n\to\infty} \underline{g}_n(x) = h(x), \quad \text{for all } x \in K.$$

By an argument similar (or rather "dual") to the one sketched above, one may prove that each of the lower envelopes $\underline{g}_n$ is pointwise limit of an ascending sequence of continuous convex functions. Thus every $\underline{g}_n$ is of class $\mathcal{C}_\sigma(K)$, and by (11), h is of class $\mathcal{C}_{\sigma\delta}(K)$.

By a similar (dual) argument one may prove h to be of class $\mathcal{C}_{\delta\sigma}(K)$.

By definition, the first Baire class is the class $\mathcal{C}_\lambda(K)$ of all pointwise limits of (not necessarily monotone) sequences from $\mathcal{C}(K)$, and by a classical lemma of Sierpinski [7, p.13], we shall have

$$h \in \mathcal{C}_{\sigma\delta}(K) \cap \mathcal{C}_{\delta\sigma}(K) = \mathcal{C}_\lambda(K). \tag{12}$$

Hence we have proved h to be of the first Baire class.

Now let $\mathcal{K}$ denote the set of all probability measures on $\overline{\partial_e K}$, and define the mappings $\rho : \mathcal{K} \to K$ and $\varphi : \mathcal{K} \to \mathbb{R}$ as follows:

$$\rho(\mu) = \int t d\mu(t) \qquad \text{(barycenter of } \mu\text{)}, \tag{13}$$

$$\varphi(\mu) = \int \tilde{f} d\mu. \tag{14}$$

A theorem of G. Choquet [3] (cf. also [6, ch.12]) states that the "barycenter formula" is valid for any affine function of the first Baire class. Applied to our function h, this means that for an arbitrary probability measure μ on K with barycenter x

$$h(x) = \int h d\mu. \tag{15}$$

If μ is a probability measure on $\overline{\partial_e K}$, then in particular:

$$h(x) = \int \tilde{f} d\mu. \tag{16}$$

Hence the mapping φ admits the factorization

$$h \circ \rho = \varphi. \tag{17}$$

Clearly ρ and φ are w^*-continuous, and K is w^*-compact. To prove h to be continuous, we consider an arbitrary closed subset F of $\mathbb{R}$. By the factorization (17) and the fact that ρ maps $\mathcal{K}$ onto K, we shall have

$$h^{-1}(F) = \rho(\varphi^{-1}(F)). \tag{18}$$

By continuity and compactness, $h^{-1}(F)$ is closed. Hence h is a continuous affine extension of f.

2) The necessity of the conditions (i),(ii) is trivial. In fact, if h is any continuous affine extension of f, then $h = \overline{f} = \underline{f}$, and

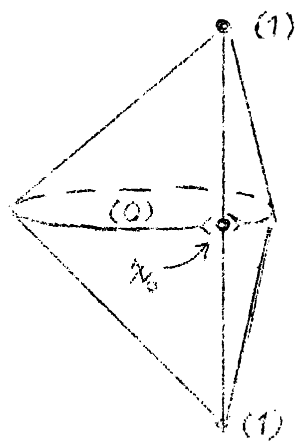
$$\int f d\nu = \int h d\nu = 0$$

for any affine dependence ν on $\partial_e K$. This completes the proof.

If $\overline{\partial_e K}$ is closed, then the condition (i) is automatically satisfied since the continuity of $\overline{f}$ and $\underline{f}$ will follow from the continuity of f by virtue of (3). (Note that (3) is independent of (i)). In this case $\overline{\partial_e K} = \partial_e K$, $\widetilde{f} = f$, and the crucial statement (16) simply reduces to the definition of h. Hence there is no more need to assume the metrizability, which in fact was used only to prove h to be in the first Baire class and thus to obtain the formulas (15) and (16).

<u>Corollary</u>. <u>Let K be a convex compact set for which $\overline{\partial_e K}$ is closed, and let f be a continuous real valued function on $\partial_e K$. The function f can be extended to a continuous affine function on K if and only if it satisfies condition (ii) of the preceding theorem</u>.

The condition (i) can not be omitted in the general case, not even if f is assumed to be uniformly continuous. In fact, it suffices to consider the well known Bourbaki example in $\mathbb{R}^3$ with boundary values as indicated in the diagram. All affine dependences on $\partial_e K$ are concentrated on the generating circle and so (ii) is satisfied, but there can be no continuous affine function with the prescribed boundary values.



Note that (i) really is violated, as $\bar{f}(x_0) = 1$ whereas $\bar{f} = f = 0$ on the rest of the generating circle.

We do not know if the conclusions of the theorem subsist without metrizability.

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